Orthogonal Subspaces

3B Two subspaces **V** and **W** of the same space $\mathbf{R}^n$ are *orthogonal* if every vector v in **V** is orthogonal to every vector w in **W**: $v^T w = 0$ for all v and w.

Example) Suppose **V** is the plane spanned by $v_1 = (1,0,0,0)$ and $v_2 = (1,1,0,0)$. If **W** is the line spanned by $w = (0, 0, 4, 5)$, is **V** orthogonal to **W**?

$$v = av_1 + bv_2 = (a+b, b, 0, 0)$$

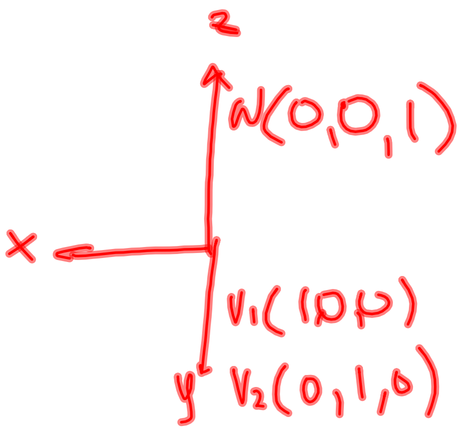
$$w = c(0,0,4,5) = \langle 0, 0, 4c, 5c)$$

$$v^T w = \begin{pmatrix} a+b \\ b \\ 0 \\ 0 \end{pmatrix} (0, 0, 4c, 5c)$$

$$= 0$$

$\therefore$ V is orth. to W.

Example 2) The floor of the classroom (extended to infinity) is a subspace V. The line where two walls meet is a subspace W (one-dimensional). The origin (0, 0, 0) is in the corner. Are they orthogonal to each other?



Example 3) Suppose **V** is still the floor but **W** is a wall (a two-dimensional space). Are they orthogonal?

x-y

(1, 2)

x - z

(5, 0, 7)

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 5 & 0 & 7 \end{bmatrix} \neq 0$$

Example 4) find the nullspace for A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 2 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

row space

$x + 2y - z = 0$

$y - 5z = 0$

$y = 5z$

$x + 10z - z = 0$

$x = -9z$

$$\begin{bmatrix} -9 \\ 5 \\ 1 \end{bmatrix} a$$

$$\begin{bmatrix} 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} -9 \\ 5 \\ 1 \end{bmatrix} = 0$$

The nullspace is orthogonal to the row space in $\mathbf{R}^n$

Find all vectors that are perpendicular to the column vectors in A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 2 & 3 & 3 \end{bmatrix}$$

Find all vectors that are orthogonal to the row space and the column space of A.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 11 \\ -2 & 3 & -1 \end{bmatrix}$$